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Journal of Geometry and Physics 49 (2004) 224–230

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

A construction of stable vector bundles on Calabi–Yau manifolds

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Received 5 March 2003

Abstract

In this paper we present a construction of stable bundles on a Calabi–Yau manifold using elementary transformation. As an application, we give examples of stable bundles on certain Calabi–Yau threefolds.

JGP SC: Differential geometry

MSC: 14J60; 14D20

Keywords: Stable vector bundle; Calabi–Yau manifold; Elementary transformation

1. Introduction

Recently holomorphic vector bundles on Calabi–Yau manifolds and their relation with D-branes have been extensively investigated. As by now quite familiar, such vector bundles are required to admit Hermitian–Yang–Mills connection for the super-symmetries to be preserved. By Donaldson–Uhlenbeck–Yau theorem, this is equivalent to the condition that the vector bundles are stable in the sense of algebraic geometry. According to Vafa's extended mirror symmetry conjecture [10], stable bundles on Calabi–Yau manifolds should correspond to special Lagrangian submanifolds of the mirror manifolds.

For Calabi–Yau manifolds with elliptic fibrations, the method of spectral cover due to Friedman–Morgan–Witten has been successfully exploited to construct stable $\mathrm{SU}(n)$-bundles [3,4]. However, the classification of stable bundles on general Calabi–Yau manifolds is largely an unexplored territory. In this paper we present another strategy for constructing stable bundles, which works for arbitrary Calabi–Yau manifolds in principle.

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0393-0440/$ – see front matter
doi:10.1016/S0393-0440(03)00089-5

Our construction is based on the notion of elementary transformation, which is defined to be the kernel of the natural map $H^0(D, L) \otimes \mathcal{O}_X \to L$ of a line bundle L on a divisor $D \subset X$. We shall show that certain minimality assumption on D guarantees the stability of the resulting bundle. This is based on a result in [7], where we started an investigation of stable sheaves on Calabi–Yau manifolds and their moduli spaces. To illustrate our method, we construct stable bundles on explicit Calabi–Yau threefolds.

2. Method of construction

Let X be a complex Calabi–Yau manifold of dimension n. In this section we recall the notion of elementary transformation to construct vector bundles on X and give a stability criterion. Let D be an effective divisor on X with the normal bundle $N_{D/X} = \mathcal{O}_D(D)$ and let $i : D \hookrightarrow X$ denote the inclusion map. Let L be a line bundle on D which is generated by global sections. Then the evaluation map $H^0(D, L) \otimes \mathcal{O}_D \to L$ is surjective and we may extend it to a surjection $H^0(D, L) \otimes \mathcal{O}_X \to i_*L$. It is known that the kernel of this map is locally free [6] and is called the elementary transformation of $H^0(D, L) \otimes \mathcal{O}_X$ along i_*L. We denote its dual bundle by $E(D, L)$. Thus we have the following exact sequence:

$$0 \to E(D, L)^\vee \to H^0(D, L) \otimes \mathcal{O}_X \to i_*L \to 0. \qquad (*)$$

Let H be an ample line bundle on X. We define the minimal H-degree $d_{\min}(H)$ as follows:

$$d_{\min}(H) = \min\{L \cdot H^{n-1} | L \in \operatorname{Pic} X, L \cdot H^{n-1} > 0\}.$$

A line bundle L is said to be H-minimal if $L \cdot H^{n-1} = d_{\min}(H)$. Such line bundles are very useful for the construction of stable bundles as shown in [7]. The following result is a special case of [7, Lemma 1.4].

Lemma 2.1. *Let (X, H) be a polarized smooth projective variety and $\mathcal{L}$ an H-minimal line bundle on D. Let Q be an H-stable torsion-free sheaf with $c_1(Q) = \mathcal{L}$ on X. Let U be a non-zero vector space and E a torsion-free sheaf which fits in the exact sequence*

$$0 \to U \otimes \mathcal{O}_X \to E \to Q \to 0.$$

E is H-stable if the map $U^\vee \to \operatorname{Ext}^1(Q, \mathcal{O}_X)$, which is obtained by applying the functor $\operatorname{Hom}(\ , \mathcal{O}_X)$ to the sequence, is injective.

The above lemma allows us to construct stable torsion-free sheaves on a given variety. The next result states that if X is a Calabi–Yau manifold, then $E(D, L)$ is a stable vector bundle.

Theorem 2.2. *Let (X, H) be a polarized Calabi–Yau manifold of dimension $n \geq 2$. Let D be a smooth irreducible effective divisor on X and L a globally generated line bundle on D. If D is H-minimal, then the bundle $E(D, L)$ is H-stable.*

Proof. By the adjunction formula, the canonical bundle of D is given by $K_D \cong \mathcal{O}_D(D)$. Hence Serre duality yields

$$H^0(D, L)^\vee \cong H^{n-1}(D, \mathcal{O}_D(D - L)) \cong \mathrm{Ext}^1(\mathcal{O}_D(D - L), \mathcal{O}_X)^\vee.$$

By applying the functor $\mathrm{Hom}_{\mathcal{O}_X}(\, , \mathcal{O}_X)$ to $(*)$, we obtain

$$0 \to \mathrm{Ext}^1(\mathcal{O}_D(D - L), \mathcal{O}_X)^\vee \otimes \mathcal{O}_X \to E(D, L) \to \mathcal{O}_D(D - L) \to 0,$$

since we have the following isomorphism, which can be checked by a local calculation:

$$\mathrm{Ext}^1_{\mathcal{O}_X}(L, \mathcal{O}_X) \cong \mathcal{O}_D(D - L).$$

Let $s \in H^0(D, L)$ be a non-zero section such that its zero scheme Z is of pure codimension two in X. Let $\mathcal{I}_Z$ denote its ideal sheaf in X. The cup product defines the natural pairing

$$\langle \, , \rangle : H^{n-1}(\mathcal{O}_D(D - L)) \times H^0(L) \to H^{n-1}(\mathcal{O}_D(D)) \cong \mathbb{C},$$

and there exists an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{I}_Z(D) \to \mathcal{O}_D(D - L) \to 0.$$

Applying $\mathrm{Hom}(\, , \mathcal{O}_X)$ to the sequence above and taking the dual of the induced cohomology sequence, we obtain the exact sequence

$$0 \to \mathrm{Ext}^1(\mathcal{I}_Z(D), \mathcal{O}_X)^\vee \to \mathrm{Ext}^1(\mathcal{O}_D(D - L), \mathcal{O}_X)^\vee \to \mathbb{C} \to 0.$$

Let $U := \mathrm{Ext}^1(\mathcal{O}_D(D - L), \mathcal{O}_X)^\vee$. The last map may be identified with $\langle \, , s\rangle$ and Ext^1 $(\mathcal{I}_Z(D), \mathcal{O}_X)^\vee$ with its kernel U_s, hence we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & & \mathcal{O}_X & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U_s \otimes \mathcal{O}_X & \xrightarrow{\alpha|_{U_s \otimes \mathcal{O}_X}} & E(D, L) & \longrightarrow & \mathcal{I}_Z(D) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & U \otimes \mathcal{O}_X & \xrightarrow{\alpha} & E(D, L) & \longrightarrow & \mathcal{O}_D(D - L) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_X & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

We have $\mathrm{Hom}(E(D, L), \mathcal{O}_X) = H^0(E(D, L)^\vee) = 0$, as can be seen by applying the functor $\mathrm{Hom}(\, , \mathcal{O}_X)$ to $(*)$. It follows that the canonical map $U_s^\vee \to \mathrm{Ext}^1(\mathcal{I}_Z(D), \mathcal{O}_X)$ is injective, hence an isomorphism. Therefore we conclude that $E(D, L)$ is stable by Lemma 2.1. □

Corollary 2.3. *Let (X, H) be a polarized Calabi–Yau threefold. Assume that there are a smooth irreducible effective divisor D which is H-minimal and a globally generated line bundle L on D. Then there exists an H-stable vector bundle E on X with the following invariants. The rank of E is $h^0(L)$ and ith Chern classes $c_1(i_*L)$* are

$$c_1(E) = [D], \qquad c_2(E) = i_*c_1(L), \qquad c_3(E) = i_*L^2,$$

and the cohomologies of E are given by

$$h^0(E) = h^0(L) + h^2(L), \qquad h^1(E) = h^1(L), \qquad h^2(E) = h^3(E) = 0.$$

Proof. Let $E(D, L)$ be as in Theorem 2.2 and let

$$c(E(D, L)) = 1 + c_1(E(D, L)) + c_2(E(D, L)) + \cdots ,$$

denote the total Chern class of $E(D, L)$. By $(*)$, we have the relation $c(E(D, L)^\vee) = c(i_*L)^{-1}$ and $c_i(i_*L)$ are given as follows:

$$c_1(i_*L) = [D], \qquad c_2(i_*L) = i_*(D^2 - L), \qquad c_3(i_*L) = i_*(D^2 - 2D \cdot L + L^2).$$

Hence we can compute $c_i(E(D, L))$ by the Grothendieck–Riemann–Roch formula

$$\mathrm{ch}(i_*L) = i_*(\mathrm{ch}(L) \cdot \mathrm{td}(N_{D/X})^{-1}),$$

where ch denote the Chern character and td the Todd class. The cohomologies of $E(D, L)$ can be computed easily from $(*)$. □

3. Stable bundles on Calabi–Yau threefolds

In this section we shall give examples of stable bundles on polarized Calabi–Yau threefolds (X, H), which appear in Corollary 3.3. For the sake of simplicity, we treat the case when L is a multiple $mH_{|D}$ of the restriction of H to D.

First we assume that X has Picard number $\rho(X) = 1$ and that its Picard group is generated by H. If there exists a smooth divisor $D \in |H|$, then D is clearly H-minimal. Assume that $L_m := mH_{|D}$ is globally generated for some $m > 0$. Then Theorem 2.2 yields an H-stable bundle E_m of rank $r_m := h^0(D, L_m)$ which fits in the exact sequence

$$0 \to E_m^\vee \to \mathcal{O}_X^{\oplus r_m} \to i_*L_m \to 0.$$

The Chern classes of E_m are given by

$$c_1(E_m) = H, \qquad c_2(E_m) = mH^2, \qquad c_3(E_m)m^2H^3.$$

The simplest example of such manifolds are quintic hypersurfaces $X \subset \mathbb{P}^4$. Let $H = \mathcal{O}_X(1)$ denote the restriction of the tautological bundle on $\mathbb{P}^4$. Then one may find smooth $D \in |H|$ and L_m is very ample, hence globally generated for all $m > 0$. Thus we obtain stable bundles E_m of rank $r_m = h^0(L_m)$, $c_1(E_m) = 1$, $c_2(E_m) = m$ and $c_3(E_m) = 5m^2$. By the two exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^4}(m-5) \to \mathcal{O}_{\mathbb{P}^4}(m) \to mH \to 0,$$

and

$$0 \to (m-1)H \to mH \to i_*L_m \to 0,$$

we compute $r_m = s_m - s_{m-1} - s_{m-5} + s_{m-6}$, where s_m is the integer defined as follows:

$$s_m = \begin{cases} \begin{pmatrix} m+4 \\ 4 \end{pmatrix} & \text{if } m \geq 0, \\ 0 & \text{if } m < 0. \end{cases}$$

To see the existence of smooth D for other Calabi–Yau threefolds with $\rho(X) = 1$, e.g. weighted complete intersections, we recall a result due to Fujita. For a polarized manifold (X, H) of dimension n, the delta genus $\Delta = \Delta(X, H)$ is defined to be the following integer:

$$\Delta = n + H^n - h^0(X, H).$$

In [5], it is proved that if $\Delta \leq 2$ and $H^n \geq 2$, then general member $D \in |H|$ is a smooth divisor. It follows that if (X, H) is a polarized Calabi–Yau threefold with $\operatorname{Pic} X \cong \mathbb{Z}[H]$, $\Delta \leq 2$ and $H^3 \geq 2$, then we may choose a smooth $D \in |H|$. Since the adjunction formula yields $K_D = H_{|D}$, D is a minimal surface of general type. It is well-known that $L_m = mK_D$ is globally generated for $m \geq 3$ under the assumption $K_D^2 \geq 2$ by a theorem of Reider [9]. Thus, for each $m \geq 3$ we obtain a stable bundle E_m of rank $r_m = h^0(mK_D)$. By Kodaira vanishing and Riemann–Roch, $h^0(mK_D)$ are calculated as follows:

$$\begin{aligned} h^0(mK_D) &= \mathcal{X}(\mathcal{O}_D(mK_D)) = \tfrac{1}{2}(m(m-1)K_D^2) + \mathcal{X}(\mathcal{O}_D) \\ &= \tfrac{1}{2}(m(m-1)H^3) + h^0(\mathcal{O}_X(H)), \end{aligned}$$

since Serre duality and Kodaira vanishing yield

$$\mathcal{X}(\mathcal{O}_D) = -\mathcal{X}(\mathcal{O}_X(-H)) = h^0(\mathcal{O}_X(H)).$$

We have the following list of weighted complete intersection Calabi–Yau threefolds with $\rho(X) = 1$, $\Delta \leq 2$ and $H^3 \geq 2$ [8, Theorem (4.1)]:

$$[1]: \quad (8) \subset \mathbb{P}(1^4, 4), \qquad [2]: \quad (4,6) \subset \mathbb{P}(1^3, 2^2, 3), \qquad [3]: \quad (6) \subset \mathbb{P}(1^4, 2),$$
$$[4]: \quad (2,6) \subset \mathbb{P}(1^5, 3).$$

Further, the invariants $(h, d) := (h^0(\mathcal{O}_X(H)), H^3)$ are given as $(h, d) = [1] : (4, 2), [2] : (3, 2), [3] : (4, 3), [4] : (5, 4)$, which computes $r_m = m(m-1)d/2 + h$ and $c_i(E_m)$ for each of the above Calabi–Yau threefolds.

Remark. In the examples above the line bundle L on D extends to a bundle on the ambient manifold X. In the case of quintic in $\mathbb{P}^4$, we may give an example of stable bundle from a globally generated L which does not extend to X, using the existence of a curve found by Voisin. As described in [1, Lemma 3.2], we may find a quintic threefold X and a smooth hyperplane section S which admits a fibration $S \to \mathbb{P}^1$ whose general fiber is a curve C of degree eight and arithmetic genus five. Since $h^0(\mathcal{O}_S(C)) = 2$ and $\mathcal{O}_S(C)$ is globally generated, we obtain a stable rank two bundle E on X with Chern classes

$$c_1(E) = 1, \qquad c_2(E) = 8,$$

which fits in the extension

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_C(1) \to 0.$$

Let $N_{C/X}$ denote the normal bundle of C in X. Since det $N_{C/X} \cong \mathcal{O}_C(1)$ lifts to $\mathcal{O}_X(1)$ and $H^2(\mathcal{O}_X(-1)) = 0$, E is also obtained from the curve C by Serre correspondence.

Let $X = \mathbb{P}^{(1,1,2,2,2)}$ [7] be the Calabi–Yau threefold which has been extensively studied in the context of mirror symmetry [2]. X is obtained as a blow-up $p : X \to \hat{X}$ of $\hat{X}$, which is a hypersurface of degree eight in the weighted projective space $\mathbb{P}^{(1,1,2,2,2)}$. X admits a structure of K3-fibration $\pi : X \to \mathbb{P}^1$ whose general fiber X_t is a quartic surface in $\mathbb{P}^3$. Let E denote the exceptional divisor of p and let F denote the class of π-fiber. It is known that the Picard group of X is generated by $H := 2F + E$ and F. Their intersection numbers are given as follows:

$$H^3 = 8, \qquad H^2 \cdot F = 4, \qquad F^2 = 0.$$

We fix an ample line bundle of the form $H_q = H + qF$ for sufficiently large q. For any divisor $D = \alpha H + \beta F$ on X, we have

$$D \cdot H_q^2 = 4((2q + 2)\alpha + \beta) \equiv 0 \pmod 4.$$

Thus we obtain $d_{\min}(H_q) = 4$ and F is H_q-minimal since $F \cdot H_q^2 = 4$.

The line bundle $H_t = H_{|X_t}$ is very ample on general fiber X_t, so $r_m = h^0(mH_t)$ can be computed for each $m > 0$ as follows:

$$r_m = \mathcal{X}(\mathcal{O}_{X_t}(mH_t)) = \frac{1}{2}(m^2 H_t^2) + 2 = 2m^2 + 2.$$

Hence for every $m > 0$ we obtain an H_q-stable bundle E_m of rank $r_m = 2m^2 + 2$ on X with the following Chern classes:

$$c_1(E_m) = F, \qquad c_2(E_m) = mH \cdot F, \qquad c_3(E_m) = 4m^2.$$

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